Double power-law in aggregation-chipping processes

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Temporal evolution of a distribution function $P(X,t)$ for X clusters is analyzed in aggregation-chipping processes, which is a model incorporating simultaneously aggregation and the chipping off of a monomeric unit from a randomly chosen aggregate. Numerical simulations show that $P(1,t)$ exhibits the singular time dependence $P(1,t)-P(1,\infty)\propto t^{-2/3}$. Using this time dependence, we find a notable double power-law distribution of $P(X,t)$ with universal exponents $-5/2$ and $-3/2$ at a sufficiently large t. In finite systems, clusters in the second power law with the exponent $-3/2$ eventually coagulate into one "monopolized" cluster. These analyses are in good agreement with the results of the simulation.

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The aggregation process is a general foundation for understanding many phenomena in both natural and social sciences. Though abundant work has been published already [1,2], the aggregation process still reveals new aspects when it is seen from different points of view. This irreversible process is represented by the Smoluchowski coagulation equation and often shows nontrivial distributions, such as the power-law distribution.

Some years ago, a model with a "chipping" process in addition to the aggregation was investigated by Krapivsky and Render [3], and Majumdar *et al.* [4] discovered the existence of a dynamical phase transition in the mass distribution of this model. The "chipping" process they investigated is one where a bit of the mass chips off at a certain rate and coagulates with the neighboring mass. This model conserves the total mass, and when the total mass exceeds a critical value, the distribution shows a drastic change which represents the appearance of the one big mass cluster in the distribution.

Furthermore, the aggregation process has been applied to the study of economics, especially the study of wealth distribution. The distribution of high-tax payers obeys, in almost every country, a power law which is widely known as the Pareto distribution [5]. Yamamoto *et al.* [6–8] proposed a simple model for the Pareto distribution, and their simulation shows a good fit with the data on chief executive officers (CEOs) in Japan and the USA.

The model considered here can be summarized as follows: initially N clusters are set where each cluster has a certain number of units, according to an initial distribution. The unit process consists of two parts, aggregation and chipping. First, two clusters chosen at random (X and Y clusters) are integrated and make one big cluster ($X+Y$ cluster). Second, one unit is chipped off from a randomly chosen cluster which has more than 2 units [$Z(\leq 2)$-cluster] and becomes a 1 cluster. In a one unit process, therefore, the total number of the clusters is conserved [X, Y, and Z clusters are changed into $(X+Y)$, $(Z-1)$, and 1 clusters]. The whole process proceeds by repeating the unit process. Note that this process also conserves the total number of units as well as the total number of clusters.

The basic equations for the probability distribution function $P(X,t)$ for X clusters are given by

$$[1-P(1,t)][1-P(1,t)]+\frac{P(2,t)}{1-P(1,t)}-[P(1,t)]^2 = \frac{dP(1,t)}{dt}\ (X=1), \tag{1}$$

$$\sum_{i+j=X} P(i,t)P(j,t)+\frac{P(X+1,t)}{1-P(1,t)}-2P(X,t)\left(1+\frac{1}{1-P(1,t)}\right) = \frac{dP(X,t)}{dt}\ (X\geq 2). \tag{2}$$

We have already reported an analysis for the steady state of this model [9]. In a previous paper, it is explained that the mean value $\langle X\rangle$ can be seen as a control parameter, and in the finite system, with $\langle X\rangle > 2$, the distribution is made of two parts, the power-law component and one "monopolized" cluster which almost has the units $(\langle X\rangle-2)\times N$. These results of the steady states are essentially the same as those of Majumdar *et al.* [4], although the total number of clusters is not kept constant in their model. For the nonsteady state, we have performed numerical simulations. The results for $P(1,t)$ are plotted in Fig. 1. It becomes evident that $P(1,t)$ exhibits singular time dependence

$$P(1,t)-P(1,\infty)\simeq t^{-2/3}. \tag{3}$$

We have also done the simulations with different initial distributions and different mean values. We have found that this behavior of $P(1,t)$ does not depend on mean values and

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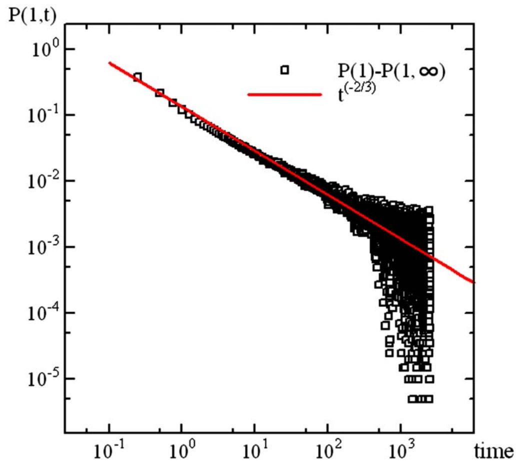

FIG. 1. (Color online) Simulation result for the time dependence of $P(1,t)$ with the number of the members $N=100\,000$ and the mean value $\langle X\rangle=3$.

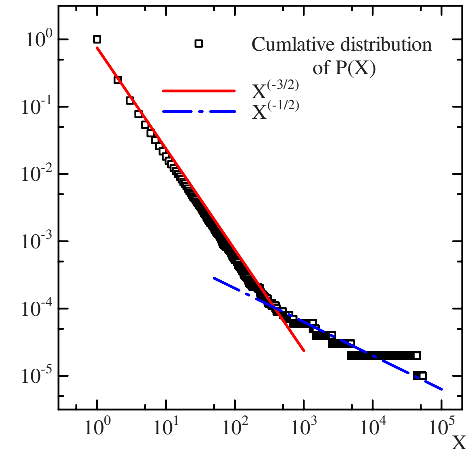

FIG. 2. (Color online) Simulation result of the cumulative distribution with the number of the members $N=100\,000$ and the mean value $\langle X\rangle=3$.

initial distributions. That is, the singular time dependence of $P(1,t)$ does not come from the singularity in initial distributions [10].

Obtaining the time dependence of $P(1,t)$ makes it possible to investigate the time dependence of the characteristic function $\phi(z,t)=\Sigma_{X=0}^{\infty}P(X,t)z^{-X}$. Equations (1) and (2) are rendered into the basic equation of $\phi(z,t)$,

$$[1-P(1,t)]\phi(z,t)^2+[z-3+2P(1,t)]\phi(z,t)-P(1,t)+\frac{1}{z}=\frac{d\phi(z,t)}{dt}. \tag{4}$$

We analyze this equation by the perturbation expansion. Let $\phi(z,t)$ be expanded as

$$\phi(z,t)=\phi_0(z)+\epsilon\phi_1(z,t), \tag{5}$$

where

$$\phi_0(z)=3-2z-2z\left(1-\frac{1}{z}\right)^{3/2} \tag{6}$$

is the characteristic function for steady state [9]. Substituting this into Eq. (4), we have the following equation in the first order of ϵ:

$$\frac{d\phi_1(z,t)}{dt}=-\phi_1(z,t)z\left(1-\frac{1}{z}\right)^{3/2}-4z^2t^{-2/3}\left[1-\frac{1}{z}+\left(1-\frac{1}{z}\right)^{3/2}\right]^2. \tag{7}$$

The solution of Eq. (7) is

$$\phi_1(z,t)=A(z,t)\exp\left[-z\left(1-\frac{1}{z}\right)^{3/2}t\right], \tag{8}$$

$$A(z,t)=\int_{t_0}^{t}\left\{-4z^2t^{-2/3}\left[1-\frac{1}{z}+\left(1-\frac{1}{z}\right)^{3/2}\right]^2\right\}\times e^{z[1-(1/z)]^{3/2}t}dt+A(z,t_0). \tag{9}$$

Now the scaling relation can be discussed. We define the scaling variable τ,

$$\tau=z\left(1-\frac{1}{z}\right)^{3/2}t, \tag{10}$$

and we find that the most singular term is

$$\phi(z,t)\simeq\phi_0(z)-4\epsilon z\left(1-\frac{1}{z}\right)^{3/2}\Psi(\tau)+A_0e^{-\tau}, \tag{11}$$

$$\Psi(\tau)=e^{-\tau}\int_{\tau_0}^{\tau}\tau'^{-2/3}e^{\tau'}d\tau'. \tag{12}$$

The scaling function $\Psi(\tau)$ obeys the inequality under the conditions $\tau,\tau_0>1$,

$$\tau^{-2/3}(1-e^{\tau_0-\tau})<\Psi(\tau)<\tau^{-2/3}-\tau_0^{-2/3}e^{\tau_0-\tau}+2\tau_0^{-1/2}e^{\tau_0-\tau}\left\{\frac{1}{2\tau_0}+\frac{1}{(2\tau_0)^2}+\frac{1}{(2\tau_0)^3}+\cdots\right\}+2\left\{\frac{1}{2\tau}+\frac{1}{(2\tau)^2}+\frac{1}{(2\tau)^3}+\cdots\right\}. \tag{13}$$

For the derivation of this inequality, see the Appendix. At a certain fixed time t_1 which is large enough, therefore, the asymptotic behavior of $\phi(z,t)$ reads

$$\phi(z,t_1)\sim\phi_0(z)-4\epsilon z\left(1-\frac{1}{z}\right)^{3/2}\left[z\left(1-\frac{1}{z}\right)^{3/2}t_1\right]^{-2/3}\sim\phi_0(z)-4\epsilon t_1^{-2/3}\left(1-\frac{1}{z}\right)^{1/2}. \tag{14}$$

Equation (14) gives us the justification for the perturbation expansion. For the calculation, we ignore the second-order term $[\phi_1(z)]^2$. From Eq. (14), the second-order term $(\phi_1)^2$ is

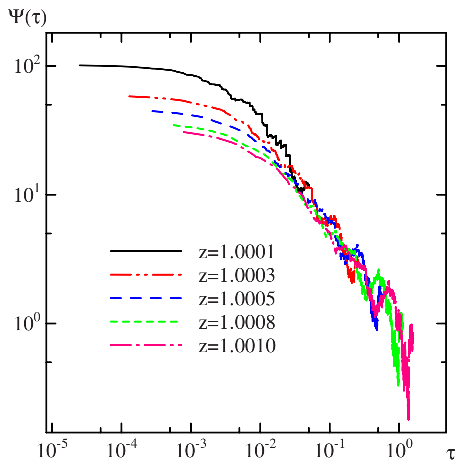

FIG. 3. (Color online) Simulation result of $\Psi(\tau)$ versus τ with the number of the members $N=100\,000$ and the mean value $\langle X\rangle=3$.

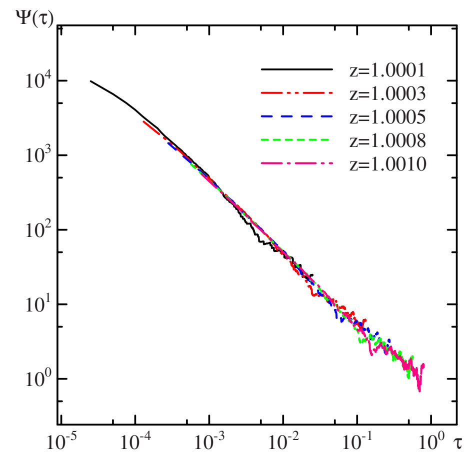

FIG. 4. (Color online) Simulation result of $\Psi(\tau)$ versus τ with the number of the members $N=100\,000$ and the mean value $\langle X\rangle=202$.

less singular than ϕ_1. This is consistent with our assumption.

It is clear that there must be some cutoff for this asymptotic behavior, otherwise the total mass represented by the second term of Eq. (14) would be infinite. From the condition $\tau>1$,

$$z\left(1-\frac{1}{z}\right)^{3/2} > \frac{1}{t}, \quad \therefore z-1 > t^{-2/3}, \tag{15}$$

near $z\sim 1$. We introduce here this condition (15) in z space which shows the existence of the cutoff in real space for the asymptotic behavior (14). For further discussion, let us compare each term of the characteristic function $P(X)z^{-X}$ between $z=1$ and $z-1=\epsilon\ll 1$. The term z^{-X} is enough small when $X>1/\epsilon$. Thus, it is difficult to see the contribution of $P(X)$ for the characteristic function over $X>X_c\sim 1/\epsilon$. Therefore this $X_c\sim\frac{1}{\epsilon}$ can be treated as the cutoff in real space, and from the condition (15) it behaves as $X_c\sim t^{2/3}$. In our simulations, the biggest cluster grows as $t^{2/3}$, therefore we believe it represents the behavior of the cutoff X_c.

What Eq. (14) signifies is that at an appropriate time and, of course, $X<X_c$, the distribution displays the double power law, the steady part of $P(X)\sim X^{-5/2}$ coming from $\phi_0(z)$, and the nonsteady part of $P(X)\sim X^{-3/2}$ coming from $(1-\frac{1}{z})^{1/2}$ in the second term of Eq. (14). As time goes by, this nonsteady $X^{1/2}$ part continues to coagulate, eventually to one cluster in finite systems. Here we show the simulation result of $P(X)$ (Fig. 2). The cumulative distribution of $P(X)$ is assuredly made by two power laws $X^{-3/2}$ and $X^{-1/2}$.

Notice that these exponents are universally observed in the aggregation processes. For the constant kernel, if the process has some injection and the total number of units grows, the exponent $-3/2$ is observed [1,10]. On the other hand, one can see the exponent $-5/2$ when the process has some conservation mechanism of the total units [11].

We have checked the scaling relation in the simulation results. Figure 3 shows the scaling function $\Psi(\tau)$ versus τ with several values of z. It is clear that the scaling relation holds when $\tau=z(1-1/z)^{3/2}t$ is large enough, i.e., when the time proceeds sufficiently. Simulations have been also carried out for different mean values $\langle X\rangle$ in the range $2.1\le\langle X\rangle\le 202$, and the scaling relation holds in the whole region (Fig. 4).

Our analysis provides two aspects. First, the time dependence of the model is elucidated. Therefore we now know the origin and the making process of the "monopolized" cluster in Refs. [4] and [9]. Second, this gives us a good possibility of comparison with experiments in natural and social sciences.

When the time t is fixed, the limit $z\to 1$ corresponds to the limit $X\to\infty$ of $P(X)$. Within this limit, though we can see only in the range $z-1>t^{(-2/3)}$, our analysis shows that the distribution is a double power law, shown in Fig. 2. In the limit $t\to\infty$ with fixed z, on the other hand, $\phi_1(z,t)\to 0$ and $P(X)\to 0$ when X is above a certain value. This state is what Majumdar *et al.* and we have shown previously. Each state has a fifty-fifty chance to be observed in experiments. That is, our results increase the opportunities of experimental verification. We believe this study helps the understanding of the deep "magic" in the aggregation systems.

APPENDIX: DERIVATION OF THE INEQUALITY (13)

When $\tau,\tau_0>1$, in the range $\tau_0<\tau'<\tau$

$$\tau'^{-2/3} > \tau^{-2/3}. \tag{A1}$$

Thus,

$$\Psi(\tau)=e^{-\tau}\int_{\tau_0}^{\tau}\tau'^{-2/3}e^{\tau'}d\tau' > e^{-\tau}\int_{\tau_0}^{\tau}\tau^{-2/3}e^{\tau'}d\tau' = \tau^{-2/3}(1-e^{\tau_0-\tau}). \tag{A2}$$

Also in that range, $\tau'^{-5/3}<\tau'^{-2/3}$, then

$$\Psi(\tau) = \tau^{-2/3} - \tau_0^{-2/3} e^{\tau_0 - \tau} + \frac{2}{3} e^{-\tau} \int_{\tau_0}^{\tau} \tau'^{-5/3} e^{\tau'} d\tau' < \tau^{-2/3} - \tau_0^{-2/3} e^{\tau_0 - \tau} + \frac{2}{3} e^{-\tau} \int_{\tau_0}^{\tau} \tau'^{-2/3} e^{\tau'} d\tau'$$

$$= \tau^{-2/3} - \tau_0^{-2/3} e^{\tau_0 - \tau} - 2(\tau^{-1/2} - \tau_0^{-1/2} e^{\tau_0 - \tau}) + 2e^{-\tau} \int_{\tau_0}^{\tau} \tau'^{-1/2} e^{\tau'} d\tau'$$

$$= \tau^{-2/3} - \tau_0^{-2/3} e^{\tau_0 - \tau} - 2(\tau^{-1/2} - \tau_0^{-1/2} e^{\tau_0 - \tau}) + 2\tau_0^{-1/2} e^{\tau_0 - \tau} \left\{ 1 + \frac{1}{2\tau_0} + \frac{1}{(2\tau_0)^2} + \frac{1}{(2\tau_0)^3} + \cdots \right\}$$

$$+ 2\left\{ 1 + \frac{1}{2\tau} + \frac{1}{(2\tau)^2} + \frac{1}{(2\tau)^3} + \cdots \right\}$$

$$= \tau^{-2/3} - \tau_0^{-2/3} e^{\tau_0 - \tau} + 2\tau_0^{-1/2} e^{\tau_0 - \tau} \left\{ \frac{1}{2\tau_0} + \frac{1}{(2\tau_0)^2} + \frac{1}{(2\tau_0)^3} + \cdots \right\} + 2\left\{ \frac{1}{2\tau} + \frac{1}{(2\tau)^2} + \frac{1}{(2\tau)^3} + \cdots \right\}. \quad \text{(A3)}$$

[1] F. Leyvraz, Phys. Rep. **383**, 95 (2003).
[2] J. Schmelzer, G. Röpke, and R. Mahnke, *Aggregation Phenomena in Complex Systems* (Wiley, New York, 1999).
[3] P. L. Krapivsky and S. Redner, Phys. Rev. E **54**, 3553 (1996).
[4] S. N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. Lett. **81**, 3691 (1998).
[5] V. Pareto, *Le Cours d'Economie Politique* (Macmillan, New York, 1897).
[6] K. Yamamoto, S. Miyazima, R. K. Koshal, M. Koshal, and Y. Yamada, Jpn. J. Ind. Appl. Math. **20**, 147 (2003).
[7] K. Yamamoto, S. Miyazima, R. K. Koshal, and Y. Yamada, Physica A **314**, 743 (2002).
[8] K. Yamamoto, S. Miyazima, H. Yamamoto, T. Ohtsuki, and A. Fujihara, *Practical Fruits of Econophysics* (Springer, Berlin, 2006), p. 349.
[9] H. Yamamoto, T. Ohtsuki, A. Fujihara, S. Tanimoto, K. Yamamoto, and S. Miyazima, Jpn. J. Ind. Appl. Math. **24**, 211 (2007).
[10] H. Takayasu, Phys. Rev. Lett. **63**, 2563 (1989).
[11] W. H. White, J. Colloid Interface Sci. **87**, 204 (1982).